Two-Way and Three-Way Pistol Duel Puzzles

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Introduction

This project focus on solving two interesting puzzles, a two-way pistol duel and a three-way pistol duel. 3 alternative approaches will be presented for solving the two-way duel,

- **geometric series approach;**
- **conditional probability approach;**
- **system of linear equations approach;**

The linear system of linear equations approach can be considered as a preparation for solving the three-way duel.

We've also developed web-based numerical simulations to verify the theoretical solutions. Please visit jjjzhang.com

Two-way Duel

Two players **A** and **B** start a pistol duel. Respectively, **A** and **B** have $0 < \alpha, \beta \leq 1$ of chances on average hitting the target. If **A** takes the first shot, what is the chance that **A** survives? What if **B** shoots first?

Solution: Let x be the probability that **A** survives if **A** shoots first and y be the probability that **B** survives if **B** shoots first.

Geometric Series Approach: When **A** shoots first **A** has a chance of α hits **B** in the 1^{st} round. If both **A** and **B** miss their first shots in the 1^{st} round then, by the Probability Multiplication Rule, **A** has a chance of $(1-\alpha)(1-\beta)\alpha$ hits **B** in the 2^{nd} round, then, a chance of $[(1-\alpha)(1-\beta)]^2\alpha$ in the 3^{rd}, so on and so forth. Thus, the total chance that **A** survives is

$$x = \sum_{j=0}^{\infty} [(1-\alpha)(1-\beta)]^j \alpha = \frac{\alpha}{1-(1-\alpha)(1-\beta)} \quad (0 \le (1-\alpha)(1-\beta) < 1).$$

Analogously, when **B** shoots first the total chance that **B** survives is

$$y = \frac{\beta}{1-(1-\alpha)(1-\beta)}.$$

That is, when **A** shoots first the chance that **A** survives is

$$x = \sum_{j=0}^{\infty} \left[(1-\alpha)(1-\beta)\right]^j \alpha = \frac{\alpha}{1-(1-\alpha)(1-\beta)},$$

and the chance that **B** survives is

$$1 - x = 1 - \frac{\alpha}{1-(1-\alpha)(1-\beta)} = \frac{(1-\alpha)\beta}{1-(1-\alpha)(1-\beta)}.$$

This solution can also be derived/explained by the other two approaches below.

Conditional Probability Approach:

Denote by U the event that player **A** hits the target;

Denote by V the event that the duel ends at the 1^{st} round;

$$P(U|V) = \frac{P(U \bigcap V)}{P(V)} = \frac{P(U)P(V|U)}{P(V)}$$
$$= \frac{\alpha \cdot 100\%}{1-(1-\alpha)(1-\beta)} = \frac{\alpha}{1-(1-\alpha)(1-\beta)}$$

That is, under the condition that the duel ends at the 1^{st} player **A**'s survival probability is

$$\frac{\alpha}{1-(1-\alpha)(1-\beta)}$$

If the duel doesn't end at the 1^{st} round it goes to the 2^{nd} round. Analogously, if the duel ends at the 2^{nd} round the conditional probability that **A** survives is still

$$\frac{\alpha}{1-(1-\alpha)(1-\beta)},$$

and so it is if the duel ends at the 3^{rd} round, the 4^{th} round, ...
Thus, in the entire duel **A**'s survival probability is

$$\frac{\alpha}{1-(1-\alpha)(1-\beta)},$$ and **B**'s survival probability is

$$1-\frac{\alpha}{1-(1-\alpha)(1-\beta)} \quad \text{or} \quad \frac{(1-\alpha)\beta}{1-(1-\alpha)(1-\beta)},$$

given that the duel process is convergent (since $0<\alpha, \beta \leq 1$).

System of Linear Equation Approach:

When **A** starts the duel and misses the first shot, the duel actually becomes being led by **B**. By the Multiplication Rule and the fact that the sum of **A**'s and **B**'s survival probabilities must be 100% either when **A** starts or **B** starts the duel, we have

$$\begin{cases} x + (1-\alpha)\,y = 1 \\ (1-\beta)\,x + y = 1 \end{cases} \implies$$

$$\begin{cases} x = \frac{\alpha}{1-(1-\alpha)(1-\beta)} \\ y = \frac{\beta}{1-(1-\alpha)(1-\beta)} \end{cases} \quad \text{Eqn. (1)}$$

For Example, $\alpha = 0.3$ and $\beta = 0.8$.

When **A** shoots first

A's survival probability is $x = \dfrac{\alpha}{1-(1-\alpha)(1-\beta)} \approx 0.3488$

B's survival probability is $1 - x$ or $(1-\alpha)\, y \approx 0.6512$

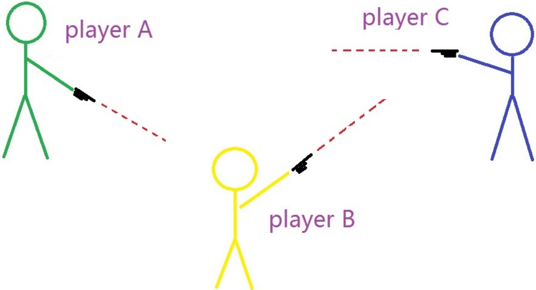
When **B** shoots first

B's survival probability is $y = \dfrac{\beta}{1-(1-\alpha)(1-\beta)} \approx 0.9302$

A's survival probability is $1 - y$ or $(1-\beta)\, x \approx 0.0698$

Three-way Duel

Part I: A, **B** and **C** take turn to shoot at each other in an order of **A, B, C, A, ...**. Respectively, **A**, **B** and **C** have α, β and γ $(0 < \alpha, \beta, \gamma \leq 1)$ of chances on average hitting the target. They choose target also in the order of **A, B, C, A, ...** unless one of them has already been eliminated. If **A** takes the first shot what are the chances that **A**, **B**, **C** survive?
Part II: What is they randomly and evenly choose target?



Part I Solution: Let the chance that **A**, **B** or **C** survives when **A**, **B** or **C** starts the duel be x, y or z, respectively.

The linear system approach used in the two-way duel can be applied to solve the three-way version by setting up 3 linear equations each based on that **A**, **B** or **C** leads the duel.

The first linear equation: **A** starts the duel. the chance that **A** survives, the chance that **B** survives and the chance that **C** survives must add up to 1 as calculated below.

When **A** starts the duel,

- The chance **A** survives is x.
- The chance that **B** survives is $(1-\alpha)y$ that is determined by the only non-zero-chance case: **A** misses.
- The chance that **C** survives is determined by the union of two exclusive non-zero-chance cases: (i) **A** hits **B** then the duel becomes two-way between **C** and **A** led by **C**, (ii) Both **A** misses their shots.

Thus, The chance that **C** survives is

$$\alpha \cdot \frac{c}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)\, z$$

The first linear equation that describes the duel led by **A** is

$$x + (1-\alpha)\,y + \alpha \cdot \frac{c}{1-(1-\alpha)\,(1-\gamma)} + (1-\alpha)\,(1-\beta)\,z = 1$$

By accordingly rotating/exchanging x, y, z, α, β and γ we have the other two linear equations that describe the duel led by **B** and the duel led **C** to form the system as

$$\begin{cases} x + (1-\alpha)\,y + \alpha \cdot \dfrac{\gamma}{1-(1-\alpha)\,(1-\gamma)} + (1-\alpha)\,(1-\beta)\,z = 1 \\[2ex] \beta \cdot \dfrac{\alpha}{1-(1-\alpha)\,(1-\beta)} + (1-\beta)\,(1-\gamma)\,x + y + (1-\beta)\,z = 1 \\[2ex] (1-\gamma)\,x + \gamma \cdot \dfrac{\beta}{1-(1-\beta)\,(1-\gamma)} + (1-\alpha)\,(1-\gamma)\,y + z = 1 \end{cases}$$

In matrix-vector form:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 - \frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1 - \frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1 - \frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix}$$

The inverse of the coefficient matrix (the augmented matrix):

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix}^{-1} =$$

$$\frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix}$$

The solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha - 1 & 0 \\ 0 & 1 & \beta - 1 \\ \gamma - 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 - \frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1 - \frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1 - \frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix}$$

For example, when $\alpha = 0.12, \beta = 0.34, \gamma = 0.56$, the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1-\frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1-\frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.1289 \\ 0.5639 \\ 0.4566 \end{bmatrix}.$$

For example, $\alpha = 0.12$, $\beta = 0.34$, $\gamma = 0.56$

When **A** leads the duel,

the chance that **A** survives is $x \approx \mathbf{0.1289}$,

the chance that **B** survives is $(1-\alpha)y \approx \mathbf{0.4962}$,

the chance that **C** survives is

$$\alpha \cdot \frac{\gamma}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)z \approx \mathbf{0.3749}.$$

Part II Solution: Let the chance that **A**, **B** or **C** survives when **A**, **B** or **C** starts the duel be x, y or z, respectively.

The Solution II used in the 2-way duel can be applied to solve the 3-way version by setting up 3 linear equations.

The first linear equation: **A** starts the duel. The sum of the chance that **A** survives, the chance that **B** survives and the chance that **C** survives must equal to 1 as calculated below.

When **A** starts the duel,

- the chance **A** survives is x.
- the chance that **B** survives is determined by the union of 3 exclusive cases:

  Case I: **A** chooses to shoot **B** and hits the target. The chance that **B** survives is 0;

  Case II: **A** chooses to shoot **C** and hits the target. The duel becomes a two-way duel between **B** and **A** led by **B**. By the results derived in **Part I** (Eqn. (1)) the chance that **B** survives is $\frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta}$;

  Case III: **A** misses the shot. The duel is still a three-way duel led by **B**. The chance that **B** survives is $(1 - \alpha)y$.

When **A** shoots first the total chance that **B** survives is
$\frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1 - \alpha)y.$

- The chance that **C** survives is determined by the union of 4 exclusive cases:

  Case I: **C** is hit by **A** or **B**. The chance that **C** survives is 0;

  Case II: **A** chooses to and hits **B**. The duel becomes two-way led by **C** against **A**. The chance that **C** survives is
  $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha + \gamma - \alpha\gamma};$

  Case III: **A** misses the shot followed by that **B** chooses to and hits **A**. The duel becomes two-way led by **C** against **B**. The chance that **C** survives is $(1 - \alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma};$

Case IV: Both **A** and **B** miss their shots. The duel is still three-way led by **C** followed by **A** and then **B**. The chance that **C** survives is $(1-\alpha)(1-\beta)z$;

When **A** shoots first the total chance that **C** survives is
$\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z.$

We have the first equation (**A** starts the duel).

$$x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z = 1.$$

By rotating/exchanging x, y, z, α, β, γ accordingly we have the other two linear equations for the duel that **B** starts the shooting followed by **C** then **A** and for the duel that **C** starts the shooting followed by **A** then **B** to form the following 3×3 linear system.

Recall the definitions, skill levels of player **A**, **B** and **C** are α, β, γ;
Chances that they survive when they start the duel are x, y, z, respectively. α, β, γ and x, y, z satisfy the linear system below.

$$\begin{cases} x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1-\alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha + \gamma - \alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma} + (1-\alpha)(1-\beta)z = 1 \\ \frac{\beta}{2} \cdot \frac{\alpha}{\alpha + \beta - \alpha\beta} + (1-\beta) \cdot \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha + \gamma - \alpha\gamma} + (1-\beta)(1-\gamma)x + y + \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma} + (1-\beta)z = 1 \\ \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha + \gamma - \alpha\gamma} + (1-\gamma)x + \frac{\gamma}{2} \cdot \frac{\beta}{\beta + \gamma - \beta\gamma} + (1-\gamma) \cdot \frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1-\alpha)(1-\gamma)y + z = 1 \end{cases}$$

In matrix-vector form:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

The inverse of the coefficient matrix (the augmented matrix):

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix}^{-1} =$$

$$\frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix}$$

The solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

Slide 26/29: Three-Way Duel 18/19 Part II

For example, $\alpha = 0.36, \beta = 0.63, \gamma = 0.27$. The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.2864 \\ 0.5418 \\ 0.2216 \end{bmatrix}.$$

$\alpha = 0.36$, $\beta = 0.63$,
$\gamma = 0.27 \Rightarrow x \approx 0.2864, y \approx 0.5418, z \approx 0.2216.$

When **A** leads the duel,

the chance that **A** survives is $x \approx \mathbf{0.2864}$,

the chance that **B** survives is
$$\frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1-\alpha)y \approx \mathbf{0.4953},$$

the chance that **C** survives is
$$\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha + \gamma - \alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma} + (1-\alpha)(1-\beta)z$$
$$\approx \mathbf{0.2183}.$$

Web-Based Numerical Simulators

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Reference

1. Class Project, Fall 2024

2. Three Way Duel or Truel at https://puzzles.nigelcoldwell.co.uk/fiftyone.htm

3. Three-Way Duel at https://www.youtube.com/watch?v=N9D4CwXTjdQ