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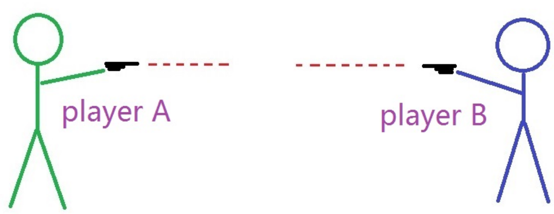
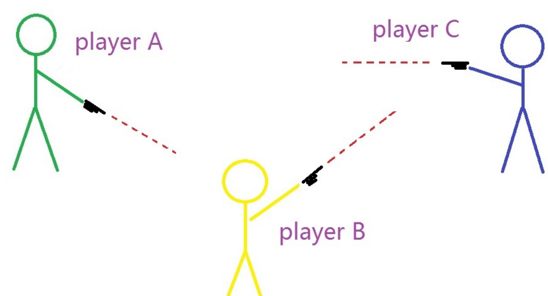
2 Acknowledgements

I thank my mother who loves board games and inspired me to choose a fun game for my project. I thank my math teacher, Mr. Caines, for encouraging my class and I to analyze deeply and to begin this research. I thank my father for providing me with advice regarding coding in Javascript and typesetting in LaTeX.

3 Problem

This project strives to solve two interesting pistol duel problems: a two-way version, and a three-way version (see pictures below). The three-way pistol duel has two variations: one where the target is randomized, and one where the targets are ordered. If each player in each game has a different skill level (e.g., 0.6 means that they will hit their target 60 percent of the time), what is each players' probability of survival (in terms of their skill level)? Does the order that they shoot in matter? To solve these problems, different approaches will be used to arrive at the correct probabilities. Then, simulations will be used to run many trials, where the players' skill levels can be customized, and verify the theoretical solutions that are presented through the different approaches.





4 Materials

- Computer
- LaTeX
- Javascript
- HTML
- WolframAlpha

5 Background Research

Prior to finalizing the approaches and proofs below, a random number generator was used to run a few trials of each duel. After understanding the skill levels' effect on the players' survival rates, and realizing that survival rates depended on which player shot first, conditional probability was first used to derive a solution for the two-way duel problem. Nigel Coldwell also provided guidance with solutions to a fun variation on his website. (A Collection of Quant Riddles With Answers, 2024). Conditional probability was learned in AP Statistics, and the implementation in this project was highly useful. After completing the conditional probability and geometric series proofs, a system of linear equations approach was attempted, but required the use of WolframAlpha to solve. In the case of the Three-Way Duel, YouTube videos were also utilized to understand the use of matrices to solve a system of linear equations. Topics such as augmented matrices, converting to matrix-vector form, and determinant matrices were previously unknown (Infinity Learn Neet, 2016). Additionally, coding in JavaScript required a guidance from YouTube videos (Bro Code, 2023), as researcher had previously only used Java.

6 Design Plan

6.1 Two-Way Duel

Problem: Two players **A** and **B** start a pistol duel. **A** and **B** have $0 < \alpha, \beta \leq 1$ chances on average of hitting their target, respectively. If **A** takes the first shot, what is the chance that **A** survives? What if **B** shoots first?

Solution: Let x be the probability that **A** survives if **A** shoots first and y be the probability that **B** survives if **B** shoots first.

6.1.1 Geometric Series Approach:

When **A** shoots first **A** has α chance of hitting **B** in the first round. If both **A** and **B** miss their shots in the first round, then, by the Probability Multiplication Rule, **A** has a chance of $(1-\alpha)(1-\beta)\alpha$ of hitting **B** in the 2^{nd} round, then, a chance of $\left[(1-\alpha)(1-\beta)\right]^2 \alpha]$ in the third, so on and so forth. Therefore, the total chance that **A** survives is

$$x = \sum_{j=0}^{\infty} \left[(1-\alpha)(1-\beta)\right]^j \alpha = \frac{\alpha}{1-(1-\alpha)(1-\beta)} \qquad (0 \leq (1-\alpha)(1-\beta) < 1).$$

Similarly, when **B** shoots first the total chance that **B** survives is

$$y = \frac{\beta}{1-(1-\alpha)(1-\beta)}$$

As stated, when **A** shoots first, the chance that **A** survives is

$$x = \sum_{j=0}^{\infty} \left[(1-\alpha)(1-\beta)\right]^j \alpha = \frac{\alpha}{1-(1-\alpha)(1-\beta)},$$

Therefore, and the chance that **B** survives is $1-x$ or $(1-\alpha)\,y$ which is

$$1 - \frac{\alpha}{1-(1-\alpha)(1-\beta)} = \frac{(1-\alpha)\beta}{1-(1-\alpha)(1-\beta)}.$$

6.1.2 Conditional Probability Approach

Let J denote the event that player **A** hits the target;

Let Z denote the event that the duel ends at the 1^{st} round;

If the duel ends in the 1^{st} round, applying conditional probability rules, the chance that **A** survives is

$$\begin{aligned} P(J \mid Z) &= \frac{P(J \bigcap Z)}{P(Z)} \\ &= \frac{P(J)P(Z \mid J)}{P(Z)} \end{aligned}$$

$$= \frac{\alpha}{1-(1-\alpha)(1-\beta)}$$

If the duel has to go the 2^{nd} round and ends in the 2^{nd} round, similarly, the conditional probability that **A** survives is still the same, and so it is if the duel ends in the 3^{rd} round, the 4^{th} round, ... Thus, in the entire duel **A**'s survival probability is

$$\frac{\alpha}{1-(1-\alpha)(1-\beta)}$$

and **B**'s survival probability is

$$1-\frac{\alpha}{1-(1-\alpha)(1-\beta)} \quad \text{or} \quad \frac{(1-\alpha)\beta}{1-(1-\alpha)(1-\beta)}$$

given that the duel process converges (since $0 < \alpha, \beta \le 1$).

6.1.3 System of Linear Equation Approach

When **A** starts the duel and misses the first shot, it is now the same thing as if the duel was led by **B**. By the Multiplication Rule and the fact that $x + y$ must be 100% either when **A** starts or **B** starts the duel,

$$\left\{\begin{array}{rcrcl} x & + & (1-\alpha)\,y & = & 1 \\ \\ (1-\beta)\,x & + & y & = & 1 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} x = \frac{\alpha}{1-(1-\alpha)(1-\beta)} \\ \\ y = \frac{\beta}{1-(1-\alpha)(1-\beta)} \end{array}\right.$$

6.1.4 Summary

Given by all three approaches, when **A** leads the duel, **A**'s survival probability is

$$x = \frac{\alpha}{1-(1-\alpha)(1-\beta)}$$

and **B**'s survival probability is

$$1 - x \text{ or } (1-\alpha)\,y = \frac{(1-\alpha)\,\beta}{1-(1-\alpha)(1-\beta)}.$$

When **B** leads the duel, **B**'s survival probability is

$$y = \frac{\beta}{1-(1-\alpha)(1-\beta)}$$

and **A**'s survival probability is

$$1 - y \text{ or } (1-\beta)\,x = \frac{(1-\beta)\,\alpha}{1-(1-\alpha)\,(1-\beta)}.$$

For example, $\alpha = 0.3$ and $\beta = 0.8$.

When **A** shoots first **A**'s survival probability is $x = \dfrac{\alpha}{1-(1-\alpha)\,(1-\beta)} \approx 0.3488$,

B's survival probability is $1 - x$ or $(1-\alpha)\,y \approx 0.6512$.

When **B** shoots first **B**'s survival probability is $y = \dfrac{\beta}{1-(1-\alpha)\,(1-\beta)} \approx 0.9302$,

A's survival probability is $1 - y$ or $(1-\beta)\,x \approx 0.0698$.

6.2 Ordered Three-Way Duel

Problem: **A**, **B** and **C** take turn to shoot at each other in an order of **A, B, C, A, ...** until two of them have been hit and there is one player left standing. Respectively, **A**, **B** and **C** have $0 < \alpha, \beta, \gamma \leq 1$ of chances on average hitting the target. To begin, **A** shoots **B**, **B** shoots **C**, **C** shoots **A**, unless one of them has already been eliminated, then the two remaining will just shoot at each other. If **A** takes the first shot what are the chances that **A**, **B**, **C** survives, respectively?

Solution: Let the chance that **A**, **B** or **C** survives when **A**, **B** or **C** starts the duel be x, y or z, respectively.

The linear system approach used in the two-way duel can be applied to solve the three-way version by setting up 3 linear equations each based on that **A**, **B** or **C** leads the duel.

The first linear equation: **A** starts the duel.

The chance that **A** survives, the chance that **B** survives and the chance that **C** survives must add up to 1 as calculated below.

- The chance **A** survives is x.
- The chance that **B** survives is $(1-\alpha)y$ determined by the chance that **A** misses.
- The chance that **C** survives is determined by the union of two exclusive cases: (1) **A** hits **B** then the duel becomes two-way between **C** and **A** led by **C**, or (2) Both **A** and B miss their shots.

Therefore, The chance that **C** survives is

$$\alpha \cdot \frac{\gamma}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)\, z$$

Therefore, The first linear equation that describes the duel led by **A** is

$$x + (1-\alpha)\, y + \alpha \cdot \frac{\gamma}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)\, z = 1$$

By accordingly rotating x, y, z, α, β and γ we have the other two linear equations that describe the duel led by **B** and the duel led **C** to form the system below

$$\begin{cases} x + (1-\alpha)\, y + \alpha \cdot \dfrac{\gamma}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)\, z = 1 \\[2em] \beta \cdot \dfrac{\alpha}{1-(1-\alpha)(1-\beta)} + (1-\beta)(1-\gamma)\, x + y + (1-\beta)\, z = 1 \\[2em] (1-\gamma)\, x + \gamma \cdot \dfrac{\beta}{1-(1-\beta)(1-\gamma)} + (1-\alpha)(1-\gamma)\, y + z = 1 \end{cases}$$

In matrix-vector form:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1 - \frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1 - \frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix}$$

The inverse of the coefficient matrix (the augmented matrix):

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix}^{-1} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1-\frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1-\frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix}$$

For example, if $\alpha = 0.12, \beta = 0.34, \gamma = 0.56$, the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{\alpha\gamma}{1-(1-\alpha)(1-\gamma)} \\ 1-\frac{\alpha\beta}{1-(1-\alpha)(1-\beta)} \\ 1-\frac{\beta\gamma}{1-(1-\beta)(1-\gamma)} \end{bmatrix} \approx \begin{bmatrix} 0.1289 \\ 0.5639 \\ 0.4566 \end{bmatrix}.$$

Meaning that when **A** leads the duel, **A**'s survival probabiity is $x \approx 0.1289$, and **B**'s survival probability is $(1-\alpha)y \approx 0.4962$, and **C**'s survival probability is $\alpha \cdot \dfrac{\gamma}{1-(1-\alpha)(1-\gamma)} + (1-\alpha)(1-\beta)z \approx 0.3749$.
If **B** leads the duel, **B** has a 0.5639 chance of surviving, and if **C** leads the duel, **C** has a 0.4566 chance of surviving.

6.3 Random Three-Way Duel

Problem: **A**, **B** and **C** take turn to shoot at each other in an order of **A, B, C, A, ...**. Respectively, **A**, **B** and **C** have α, β and γ $(0 < \alpha, \beta, \gamma \leq 1)$ of chances on average hitting the target. They choose the target that they will shoot at randomly, unless one of the other two players has already been eliminated (in that case, it becomes a two-way duel). If **A** takes the first shot what are the chances that **A**, **B**, **C** survive, respectively?

Solution: Let the chance that **A**, **B** or **C** survive when **A**, **B** or **C** starts the duel be x, y or z, respectively. Similar to the ordered version, this random three-way duel will be solved by setting up 3 linear equations.

The first linear equation: **A** starts the duel.

- **A**'s survival probability is x.
- **B**'s survival probability is determined by the union of two exclusive cases:

  Case 1. **A** chooses to shoot **C** and successfully hits. The duel becomes a two-way duel between **B** and **A** led by **B**. **B**'s survival probability is $\frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta}$

  Case 2: **A** misses their shot. The duel is still a three-way duel led by **B**. In this case. **B**'s survival probability is $(1 - \alpha)y$.

  Therefore, when **A** shoots first total **B**'s survival probability is $\frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1 - \alpha)y$.
- **C**'s survival probability is determined by the union of three exclusive cases:

  Case 1: **A** chooses to and successfully hits **B**. The duel becomes two-way between **C** and **A** led by **C**. **C**'s survival probability is $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha + \gamma - \alpha\gamma}$;

  Case 2: **A** misses the shot, followed by **B**, who chooses to and successfully hits **A**. The duel becomes two-way between **C** and **B** led by **C**. **C**'s survival probability is $(1 - \alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma}$;

  Case 3: Both **A** and **B** miss their shots. The duel is still three-way led by **C** followed by **A** and then **B**. **C**'s survival probability is $(1 - \alpha)(1 - \beta)z$;

  Therefore, when **A** shoots first total **C**'s survival probability is

  $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1 - \alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1 - \alpha)(1 - \beta)z$.

We have the first equation that describes the duel led by **A**,

$$x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha + \beta - \alpha\beta} + (1 - \alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha + \gamma - \alpha\gamma} + (1 - \alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta + \gamma - \beta\gamma} + (1 - \alpha)(1 - \beta)z = 1.$$

By accordingly rotating x, y, z, α, β, γ we have the other two linear equations for the duel led by **B** and the duel led by **C** to form a 3×3 linear system below. Recall the definitions of x, y, z, this system will find the probabilities that **A**, **B** or **C** survives when the duel is led by each of them respectively.

$$\begin{cases} x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z = 1 \\ \\ \frac{\beta}{2} \cdot \frac{\alpha}{\alpha+\beta-\alpha\beta} + (1-\beta) \cdot \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha+\gamma-\alpha\gamma} + (1-\beta)(1-\gamma)x + y + \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\beta)z = 1 \\ \\ \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha+\gamma-\alpha\gamma} + (1-\gamma)x + \frac{\gamma}{2} \cdot \frac{\beta}{\beta+\gamma-\beta\gamma} + (1-\gamma) \cdot \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)(1-\gamma)y + z = 1 \end{cases}$$

In matrix-vector form:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta} + \frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma} + \frac{\beta\gamma}{\beta+\gamma-\beta\gamma} - \frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1 - \frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta} + \frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma} + \frac{\beta\gamma}{\beta+\gamma-\beta\gamma} - \frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1 - \frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta} + \frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma} + \frac{\beta\gamma}{\beta+\gamma-\beta\gamma} - \frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

The inverse of the coefficient matrix (the augmented matrix, same as the one from the ordered version):

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix}^{-1} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix}$$

The solution:

$$\begin{bmatrix} x \\ \\ y \\ \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

For example, $\alpha = 0.36, \beta = 0.63, \gamma = 0.27$. The solution is $x \approx 0.2864, y \approx 0.5418, z \approx 0.2216$.

The solutions to the matrix are the player's survival probabilities **if** they are the ones starting the duel. Meaning that when **A** starts the duel **A**'s survival probability is $x \approx 0.2864$; When **B** starts the duel **B**'s survival probability is $y \approx 0.5418$; **C** starts the duel **C**'s survival probability is $z \approx 0.2216$;

Therefore, when **A** leads the duel, the probabilities that **A**, **B** or **C** survives are:

A: $x \approx 0.2864$;

B: $\frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y \approx 0.4953$;

C: $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z \approx 0.2183$.

Survival probabilities in the duel led by **B** and the duel led by **C** can be similarly computed using the derived equations on page 8.

7 Simulators - Final Design

At jjjzhang.com, the three different simulators can be found: the two-way duel, the three-way ordered duel, and the three-way randomized duel. These simulations verify the mathematical results derived above, as well as conduct the theoretical results of many consecutive games.

On the webpage, a user can input the skill levels α, β, and γ for players **A**, **B**, and **C**, repectively. Math.random method generates a value in $(0, 1)$ and if the random value is less than α, β, or γ, respectively, the shot will be successful. While-loops and for-loops are used to run until there is one survivor. The solutions (WolframAlpha) derived above were entered into the program to quickly calculate a player's theoretical win rate in one click.The webpage can also run multiple trials of rounds with the same skill levels and calculate the experimental chances of survival for each player. With more trials, the rate of survival for each player converges to the theoretical rate of survival.

The website is developed in HTML and JavaScript.

While using the formulas for the players' survival probabilities provide definite theoretical survival rates, the coded simulators provide real-world applications, where probability is not always completely randomized. Thus, though the coded simulators may take more time to run more trials, they allow insight into experimental probability.

8 Results and Conclusion

Formulas have been derived to calculate each player's theoretical survival rate when placed into a game where they are given a certain shooting-skill average and have to shoot each other (either two players or three). The systems of linear equations approach was used to solve all 3 variations, and the use of matrices was introduced. The final design was the Javascript simulators, where one or multiple trials of a game with different conditions can be trialed to find theoretical and experimental survival rates.

8.1 Future

- What about in 4,5,6,... player games?
- Can this be applied to sports, like basketball or soccer where it's also important to shoot on target?
- Can players' skill levels change as the duel goes on?

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