Train Linking Puzzle and Pistol Duel Puzzle. -- Game Theory Proof

Preprint · October 2024

DOI: 10.13140/RG.2.2.26759.82089

1 author:

Jasmine Zhang
City Colleges of Chicago Richard J. Daley College

Train Linking Puzzle and Pistol Duel Puzzle

Jasmine Zhang, Walter Payton College Prep, Chicago

October, 2024

Introduction This exercise report includes two interesting puzzles, a train linking puzzle and a pistol duel puzzle. For the duel puzzle we first present solution to a two-way duel as a preparation for solving the three-way version.

It is amazing if a train just vanishes without any reasonable explanation. On an infinitely long single railroad track a fast train catches a slow train then they link up to be one train and move at the speed of the slow train. Assume that there were n trains of n different speeds placed in a certain order by their speeds on this railroad track and start moving at the same time. After a long enough time has elapsed some trains may vanish. How many trains will remain on the track? Considering all possible permutation of the train initial placement orders what is the average number of remaining trains? For example, there are initially two trains. If the slow train is placed behind the fast train, there will remain 2 trains on the track. If the slow train is placed in front of the fast train, they will become 1 train. The average of remaining trains is 1.5. What about there are initially n trains? This seems like a classical puzzle, however, doesn't seem popular on the internet. We found a page of discussion but it doesn't provide a clear solution/answer (Ref. 1).

The pistol duel puzzles are obviously classical. 2 players of certain shooting skill level (average chance of hitting target) take turn to shoot each other. What is the chance that each of them will survive? It is a more challenging puzzle if the duel starts with 3 players each randomly chooses one of the other two players to shoot (Ref. 2).

Abstract In this exercise report, for the train linking puzzle, we present a formulation that calculates the average number of remaining train with a proof by induction.

For the pistol duel puzzles, the solution or discussion that we've found on the internet assumes certain numerical valued skill level, $\frac{1}{3}, 0.5$ or 100%, etc., and uses a natural approach of geometric series. We generalize the skill levels to be parametric, α, β and γ. We've worked out a solution by system of linear equations which is more effective than the geometric series approach. We see the advantage when apply the linear system approach to the three-way duel. To verify the theoretical solution we conduct a numerical experiment in MATLAB© for the three-way duel puzzle. Copy of codes are included at the end of this report.

Train Linking Puzzle: n ($n = 1, 2, 3, ...$) trains are appropriately spaced on a long enough single railroad track and start moving at the same time at n different speeds. If a train catches up to the one in front of it, they link up to form one train moving at the speed of the slower train. Eventually, how many trains in average will remain on the railway considering that the trains are initially placed in any permutation of orders by their speeds?

Solution: Let $n \in \{1, 2, 3, ...\}$ be the number of initial trains. Let A_n be the average number of remaining trains. A_n can be computed as below, followed by a proof by induction.

$$A_n = \sum_{j=1}^{n} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}.$$

Proof By Induction:

Initial Case: $A_1 = 1$ is trivially true.

Inductive Case: Assume that $A_n = \sum_{j=1}^{n} \frac{1}{j}$ is true for $n = k \geq 1$, i.e., $A_k = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k}$.

We are to show that $A_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} + \frac{1}{k+1}$.

The first k trains are appropriately spaced and placed on the track. That leaves $k+1$ possible spots for the $(k+1)^{th}$ to join. Without loss of generality, we assume that the $(k+1)^{th}$ train is faster than the other k train. No matter how the other k trains are placed in terms of their speeds, when and only when the $(k+1)^{th}$ train is placed at the leading spot (in front of the other trains) the $(k+1)^{th}$ train increases the number of remaining trains by 1, otherwise, it always vanishes in one of the first k trains and doesn't affect the train linking result. Taking a weighted average, A_{k+1} must be

$$A_{k+1} = A_k + \frac{1}{k+1} \cdot 1 = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{k} + \frac{1}{k+1} = \sum_{j=1}^{k+1} \frac{1}{j} \qquad \text{End of Proof.}$$

Remark: Interestingly, the solution is a partial sum of the harmonic series thus $\lim_{n \to \infty} A_n \to \infty$. That is, infinitely many trains will result infinitely many remaining trains in average.

Pistol Duel Puzzle (Two-Way and Three-Way)

Part I (Two-Way): Two players **A** and **B** start a pistol duel. Respectively, **A** and **B** have α and β of chances on average hitting the target for $0 < \alpha, \beta \leq 1$ (i.e., $0 \leq (1-\alpha)(1-\beta) < 1$). If **A** takes the first shot, what is the chance that **A** survives? What if **B** shoots first?

Part I Solution:

Let x be the chance that **A** survives if **A** shoots first and y be the chance that **B** survives if **B** shoots first.

A Natural Approach: When **A** shoots first **A** has a chance of α hits **B** in the 1^{st} round. If both **A** and **B** miss their first shots in the first round then, by the Probability Multiplication Rule, **A** has a chance of $(1-\alpha)(1-\beta)\alpha$ hits **B** in the 2^{nd} round, then, a chance of $[(1-\alpha)(1-\beta)]^2 \alpha$ in the 3^{rd}, so on and so forth. Thus, the total chance that **A** survives is

$$x = \sum_{j=0}^{\infty} [(1-\alpha)(1-\beta)]^j \alpha = \frac{\alpha}{\alpha + \beta - \alpha\beta} \quad (\text{for } 0 \leq (1-\alpha)(1-\beta) < 1).$$

Similarly, when **B** starts the duel the total chance that **B** survives is

$$y = \sum_{j=0}^{\infty} [(1-\alpha)(1-\beta)]^j \beta = \frac{\beta}{\alpha + \beta - \alpha\beta}.$$

An Alternative Approach: When **A** starts the duel and misses the first shot, the duel actually becomes the same duel with **B** taking the first shot. By the fact that the sum of the chances that **A** survives or **B** survives must be 100% either when **A** starts the duel or **B** starts the duel, we have

$$\left\{ \begin{array}{rcrcl} x & + & (1-\alpha)y & = & 1 \\ \\ (1-\beta)x & + & y & = & 1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} x = \frac{\alpha}{\alpha+\beta-\alpha\beta} \\ \\ y = \frac{\beta}{\alpha+\beta-\alpha\beta} \end{array} \right. \qquad \text{Eqn. (1)}$$

Part II (Three-Way): 3 players **A**, **B** and **C** start a pistol duel. Respectively, **A**, **B** and **C** have α, β and γ of chances on average hitting the target for $0 < \alpha, \beta, \gamma \leq 1$. When each of **A**, **B** and **C** shoots they evenly choose at random one of the two targets unless one of them has already been eliminated. If **A** takes the first shot followed by **B** and then **C**, what is the chance that **A** survives? the chance that **B** survives? the chance that **C** survives?

Part II Solution:

Let the chance that **A**, **B** or **C** survives when **A**, **B** or **C** starts the duel be x, y or z, respectively.

The alternative approach used in **Part I** can be applied to solve **Part II** by setting up 3 linear equations.

The first linear equation: **A** starts the duel. The sum of the chance that **A** survives, the chance that **B** survives and the chance that **C** survives must equal to 1 as calculated below.

- The chance **A** survives is x.

- The chance that **B** survives is determined by the union of 3 exclusive cases:

  Case I: **A** chooses to shoot **B** and hits the target. The chance that **B** survives is 0;

  Case II: **A** chooses to shoot **C** and hits the target. The duel becomes a two-way duel between **B** and **A** led by **B**. By the results derived in **Part I** (Eqn. (1)) the chance that **B** survives is $\dfrac{\alpha}{2} \cdot \dfrac{\beta}{\alpha + \beta - \alpha\beta}$;

  Case III: **A** misses the shot. The duel is still a three-way duel led by **B**. The chance that **B** survives is $(1 - \alpha)y$.

  When **A** shoots first the total chance that **B** survives is $\dfrac{\alpha}{2} \cdot \dfrac{\beta}{\alpha + \beta - \alpha\beta} + (1 - \alpha)y$.

- The chance that **C** survives is determined by the union of 4 exclusive cases:

  Case I: **C** is hit by **A** or **B**. The chance that **C** survives is 0;

  Case II: **A** chooses to and hits **B**. The duel becomes two-way led by **C** against **A**. The chance that **C** survives is $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma}$;

  Case III: **A** misses the shot followed by that **B** chooses to and hits **A**. The duel becomes two-way led by **C** against **B**. The chance that **C** survives is $(1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma}$;

  Case IV: Both **A** and **B** miss their shots. The duel is still three-way led by **C** followed by **A** and then **B**. The chance that **C** survives is $(1-\alpha)(1-\beta)z$;

  When **A** shoots first the total chance that **C** survives is $\frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z$.

We have the first equation (**A** starts the duel).

$$x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z = 1.$$

By rotating/exchanging x, y, z, α, β, γ accordingly we have the other two linear equations for the duel that **B** starts the shooting followed by **C** then **A** and for the duel that **C** starts the shooting followed by **A** then **B** to form the following 3×3 linear system.

$$\begin{cases} x + \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y + \frac{\alpha}{2} \cdot \frac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z = 1 \\ \frac{\beta}{2} \cdot \frac{\alpha}{\alpha+\beta-\alpha\beta} + (1-\beta) \cdot \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha+\gamma-\alpha\gamma} + (1-\beta)(1-\gamma)x + y + \frac{\beta}{2} \cdot \frac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\beta)z = 1 \\ \frac{\gamma}{2} \cdot \frac{\alpha}{\alpha+\gamma-\alpha\gamma} + (1-\gamma)x + \frac{\gamma}{2} \cdot \frac{\beta}{\beta+\gamma-\beta\gamma} + (1-\gamma) \cdot \frac{\alpha}{2} \cdot \frac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)(1-\gamma)y + z = 1 \end{cases}$$

In matrix-vector form:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

The inverse of the coefficient matrix (the augmented matrix):

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha)(1-\beta) \\ (1-\beta)(1-\gamma) & 1 & (1-\beta) \\ (1-\gamma) & (1-\alpha)(1-\gamma) & 1 \end{bmatrix}^{-1} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix}$$

The solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

For example, when $\alpha = 0.36, \quad \beta = 0.63, \quad \gamma = 0.27$, the solution is

$$\begin{bmatrix} x \\ \\ y \\ \\ z \end{bmatrix} = \frac{1}{1-(1-\alpha)(1-\beta)(1-\gamma)} \begin{bmatrix} 1 & \alpha-1 & 0 \\ \\ 0 & 1 & \beta-1 \\ \gamma-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\beta+\gamma-\beta\gamma}\right) \\ \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\gamma-\alpha\gamma}\right) \\ \\ 1-\frac{1}{2}\left(\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}+\frac{\alpha\gamma}{\alpha+\gamma-\alpha\gamma}+\frac{\beta\gamma}{\beta+\gamma-\beta\gamma}-\frac{\alpha\beta\gamma}{\alpha+\beta-\alpha\beta}\right) \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.2864 \\ \\ 0.5418 \\ \\ 0.2216 \end{bmatrix}.$$

That is, as explained above (page 3), when **A** leads the duel,

the chance that **A** survives is $x \approx \mathbf{0.2864}$,

the chance that **B** survives is $\dfrac{\alpha}{2} \cdot \dfrac{\beta}{\alpha+\beta-\alpha\beta} + (1-\alpha)y \approx \mathbf{0.4953}$,

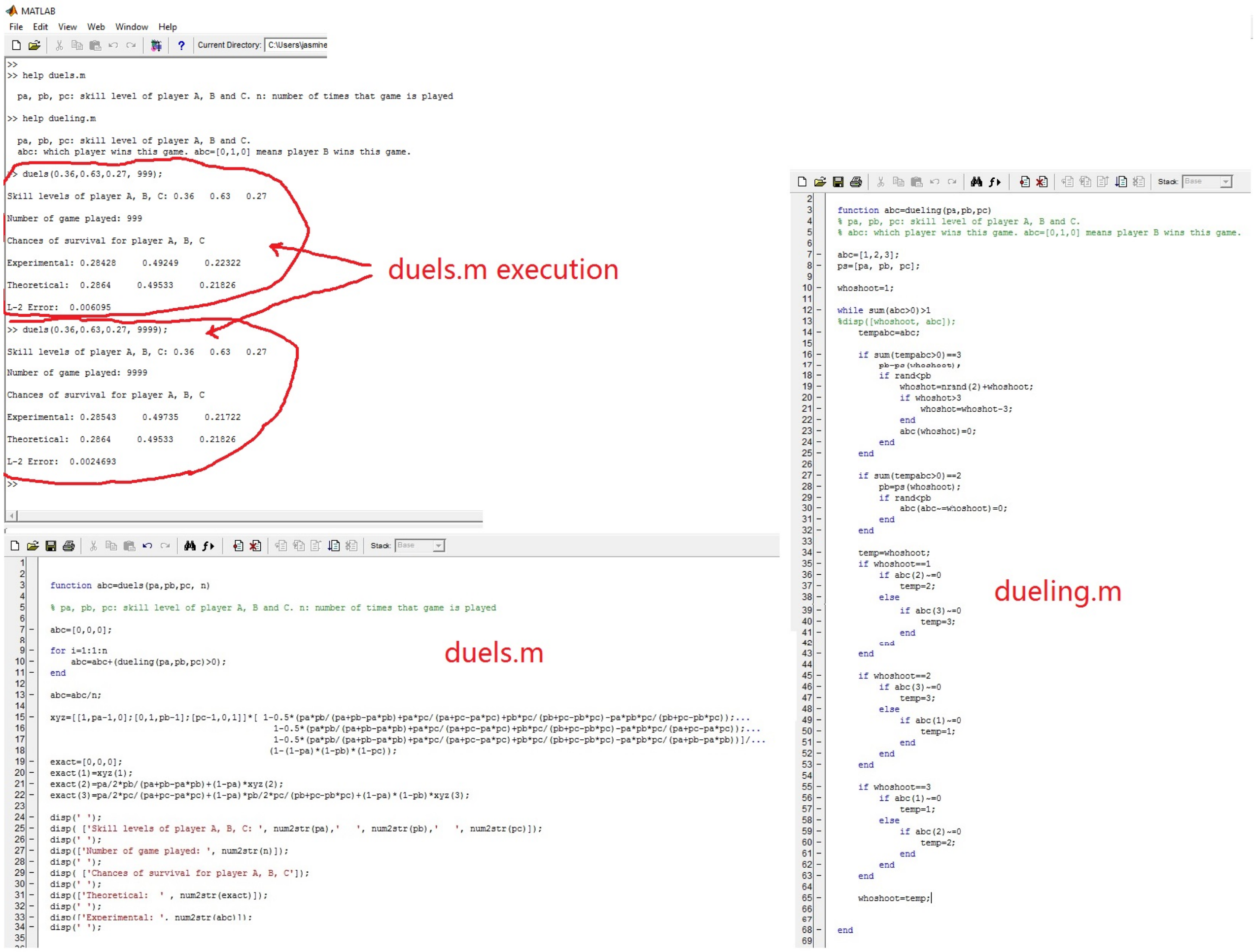
the chance that **C** survives is $\dfrac{\alpha}{2} \cdot \dfrac{\gamma}{\alpha+\gamma-\alpha\gamma} + (1-\alpha) \cdot \dfrac{\beta}{2} \cdot \dfrac{\gamma}{\beta+\gamma-\beta\gamma} + (1-\alpha)(1-\beta)z \approx \mathbf{0.2183}$.

Answers for a duel led by **B** or **C** can be similarly computed.

We've conducted a numerical simulation in MATLAB© to verify this theoretical solution. Copies of the MATLAB© codes and execution outputs (to verify the numbers in red) are presented in next page.

A challenge question: With parametric skill levels α, β and γ, instead of choosing a target at random, what if each of **A**, **B** and **C** always choose the optimal target to maximize their own chances of survival, when shooting at sky is allowed or not allowed? There are some discussion on internet but they are based on given numerical valued skill levels (Ref. 3).

MATLAB© numerical experiment for **Pistol Duel Puzzle Part II (three-way)**:



Reference

1. Critical thinking rail track problem at https://mathoverflow.net/questions/385370/critical-thinking-rail-track-problem

2. Three Way Duel or Truel at https://puzzles.nigelcoldwell.co.uk/fiftyone.htm

3. Three-Way Duel at https://www.youtube.com/watch?v=N9D4CwXTjdQ