Interesting Generalized Airplane Seating Puzzle -- Number Theory and Combinatorics

Preprint · May 2024

DOI: 10.13140/RG.2.2.14455.84645

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Generalized Airplane Seating Puzzle

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May, 2024

Introduction Assume you are the first person among 100 passengers to board a flight. You reach for your boarding pass, and you can't find it. You decide to take a random seat out of the 100 seats on the plane since you don't remember which seat was assigned on your boarding pass. Clearly, there is a $\frac{99}{100}$ chance that you will pick another passenger's seat. The following 99 passengers have their boarding pass, and they board in the following manner: If their assigned seats are not taken, they will sit in their assigned seats. If their assigned seats are taken, they will sit in random seats. What is the probability that the 100^{th} person to board the plane will get to sit in his assigned seat? What if there are 10000 instead of 100 passengers? Moreover, what is the probability for each passenger to sit in his assigned seat?

Abstract This is a retelling of a rather popular puzzle on the internet that may follow the title of "Airplane Seating Problem" (webpage 1; webpage 2). Various discourses and solutions may be found on the internet. In this work, we are to generalize this puzzle and present a solution with numerical experiment. Up to what we have searched in public resources we haven't found this generalized version or other versions of the puzzle.

The Original Airplane Seating Puzzle: *100 passengers are in line, boarding an airplane with 100 seats, one at a time. The first passenger has lost his boarding pass, so he sits in a random seat. The second passenger and following passengers will take their assigned seats if their assigned seats are not occupied. Otherwise, they will randomly select seats from unoccupied seats. What is the probability that the last passenger (the* 100^{th} *passenger to board) will get to sit in his assigned seat?*

The answer is $\frac{1}{2}$. The explanation may be rather straightforward. There are two important seats, the first passenger's and the last passenger's assigned seats. Before the last passenger gets to sit some passengers, at least the first passenger and at most all of the first 99 passengers, must randomly select seats. Each of these random seat selections is independent. The random process ends as soon as one of the two important seats is randomly selected. If the first passenger's assigned seat is selected, the last passenger is guaranteed to have his assigned seat; If the last passenger's assigned seat is selected, the last passenger is guaranteed to lose his assigned seat. The probabilities that these two cases occur are $50-50$, because the two important seats have even chance to be picked by all of those who get to select seat at random from the available unoccupied seats.

Remarks:

- There may be some passengers who don't need to randomly select seats. Those passengers and their seats make no effect to the probability that the last passenger will get to sit in his assigned seat. This interesting random process is just like tossing a coin. The coin has equal chance to land with the head facing up or the tail facing up. In the case that the coin lands standing, not head nor tail, the coin is tossed again.
- From the discussion above it is clear that when there are two or more passengers in the line the last passenger always has $\frac{1}{2}$ of chance to sit in his own seat. That is, the number "100" can be generalized to any positive integer that is greater than 2.

Now, let's take a look at the probabilities that the other passengers get to sit in their assigned seats.

A Generalized Airplane Seating Puzzle: *n ($n = 1, 2, 3, ...$) passengers are in line, boarding an airplane with n seats, one at a time. The first passenger has lost his boarding pass, so he sits in a random seat. The second passenger and the following passengers will take their assigned seats if their assigned seats are not occupied. Otherwise, they will randomly select seats from the unoccupied seats. What is the probability that the m^{th} ($m = 1, 2, 3, ...n$) passenger in the line will get to sit in his assigned seat?*

Solution We first present the solution and it follows that the proof and explanation.

Let $Prob(n, m)$, for $n = 1, 2, 3, ...$ and $m = 1, 2, 3, ..., n$, be the probability that the m^{th} passenger in the line of n passengers gets to sit in his assigned seat. $Prob(n, m)$ satisfies the following.

For $m = 1$, $Prob(n, 1) = \frac{1}{n}$; For $m = 2, 3, ..., n$, $Prob(n, m) = \frac{n-m+1}{n-m+2}$.

$Prob(n, m)$ can be visualized in Table 1 below.

Table 1: Probability that the m^{th} passenger in the line of n passengers gets the assigned seat.

n \ m	1	2	3	4	5	6	7	8	...
1	1								
2	1/2	1/2							
3	1/3	2/3	1/2						
4	1/4	3/4	2/3	1/2					
5	1/5	4/5	3/4	2/3	1/2				
6	1/6	5/6	4/5	3/4	2/3	1/2			
7	1/7	6/7	5/6	4/5	3/4	2/3	1/2		
8	1/8	7/8	6/7	5/6	4/5	3/4	2/3	1/2	
$\vdots$									$\ddots$

As seen in Table 1 $Prob(n, m)$ forms a simple pattern. For example, the 4^{th} passenger in line of total 5 passengers has a $\frac{2}{3}$ chance to sit in his assigned seat.

We have two versions of proof for $Prob(n, m)$ stated above, Proof 1: similar enough to the explanation for the original puzzle; Proof 2: by induction.

Proof 1 (similar to that for the solution of the original puzzle).

For $m = 1$, that $Prob(n, 1) = \frac{1}{n}$ (the $m = 1$ column of Table 1) is clearly true.

Denote, respectively, the assigned seats of all passengers in the line by $s_1, s_2, s_3, ..., s_m, s_{m+1}, ..., s_n$.

For $m = 2, 3, ..., n$, again, similar to the explanation for the original puzzle, there are two important SETS of seats. They are, Set 1 $= \{s_m\}$ and Set 2 $= \{s_1, s_m, s_{m+1}, ..., s_n\}$ (total $n - m + 1$ seats). Again, before the m^{th} passenger in the line gets to sit there is a random process when some passenger(s) must randomly select set(s). Seats in Set 1 or Set 2 get to be selected with probabilities in the ratio $1 : n - m + 1$. As soon as the only seat s_m in Set 1 is selected the random process ends and the m^{th} passenger is guaranteed to lose his assigned seat. As soon as one of the seats in Set 2 is picked the random process ends and the m^{th} passenger is guaranteed to sit in his assigned seat. That is, the m^{th} $(n \geq m \geq 2)$ passenger has a

$$\frac{n - m + 1}{1 + n - m + 1} = \frac{n - m + 1}{n - m + 2}$$ chance to sit in his assigned seat.

In order words, he has a $\dfrac{1}{n-m+2}$ chance to lose his assigned seat. Thus,

for $m = 1$, $Prob(n, 1) = \dfrac{1}{n}$; for $m = 2, 3, ..., n$, $Prob(n, m) = \dfrac{n-m+1}{n-m+2}$. End of Proof 1.

Proof 2 (by induction).

For convenience, in this part of discussion we focus on Table 1.

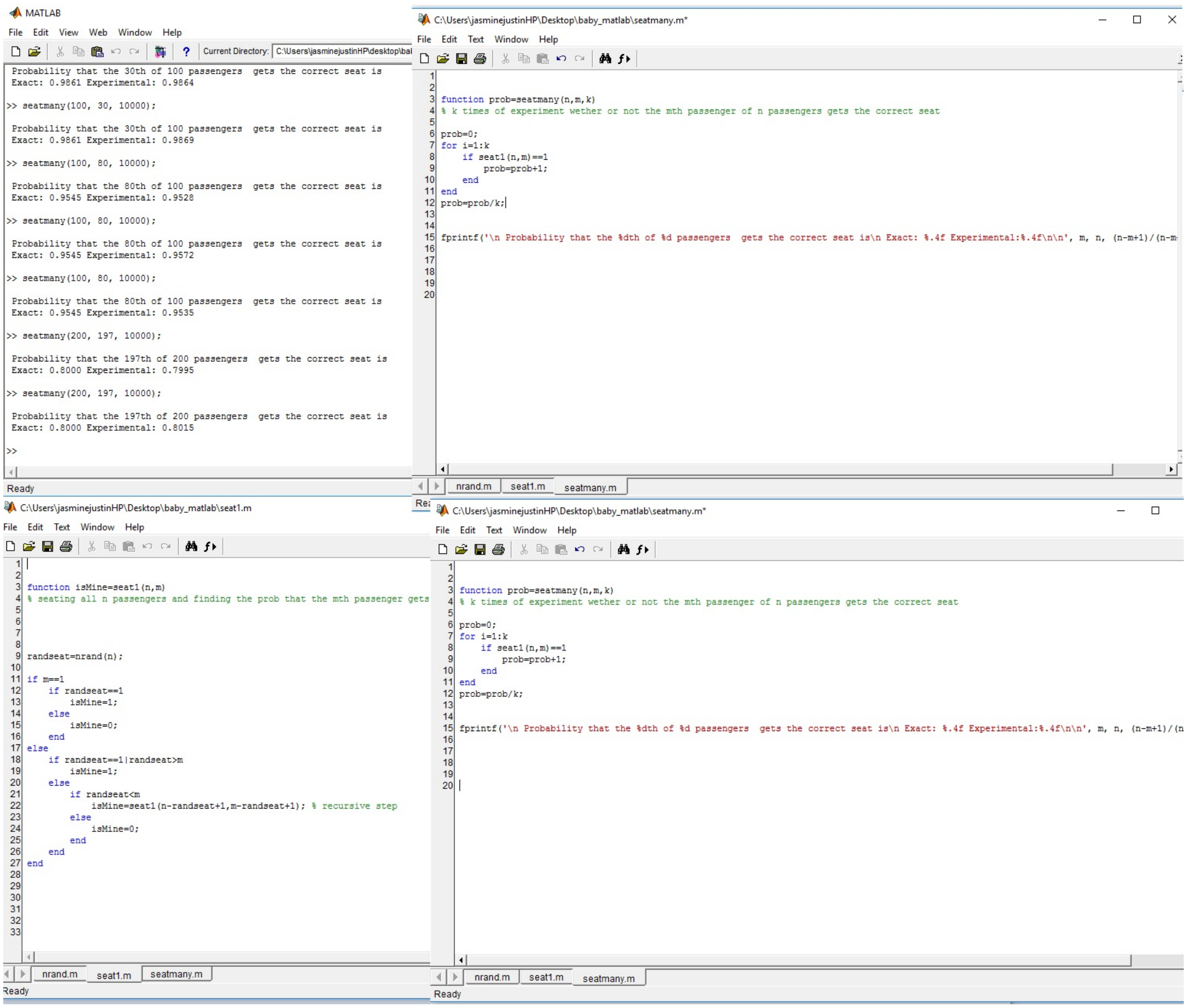
The initial cases (the first 3 rows of Table 1, $n = 1,\ 2,\ 3$) can easily be calculated and confirmed using basic probability concepts and techniques such as multiplication rules, and the first cell of all rows are clearly true.

For the inductive step, assume that Table 1 is true for $n = k \geq 3$. Without loss of generality, a passenger with boarding pass is added to the second position of the passenger line. When the first passenger randomly selects a seat it leads to two and only two exclusive cases, the first passenger selects the seat of the newly added passenger (in the second spot of the line) or other seat. If the first passenger picks the seat of the newly added passenger's (the probability is $\frac{1}{k+1}$) then the first passenger no longer makes effect to the rest of the random process and the newly added passenger becomes the only passenger who doesn't have boarding pass and starts the random process. If the first passenger picks other seat then the newly added passenger keeps his assigned seat (the probability is $\frac{k}{k+1}$) and makes no effect to the entire random process. That is, after the first passenger selects a seat the puzzle of size $k + 1$ is reduced to an identical puzzle of size k. That is, except for the first and the second passenger the other passengers in the line keep the same probabilities of getting their assigned seats as before the new passenger is added. This explains the constant diagonal property of Table 1 and concludes the prove. That is,

For $m = 1$, $Prob(n, 1) = \dfrac{1}{n}$; For $m = 2, 3, ..., n$, $Prob(n, m) = \dfrac{n-m+1}{n-m+2}$. End of Proof 2.

Numerical Experiment

We have realized that this puzzle is also an excellent computer programming exercise. The theoretical solution (contents of Table 1) may be verified by numerical simulation. We include our program coded in MatLab in this work. The random process during the passenger seating is accomplished by a recursive function as shown in the copies below.



Reference

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